

RIESZ TRANSFORMS ON Q-TYPE SPACES WITH APPLICATION TO QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. In this paper, we prove the boundedness of Riesz transforms $\partial_j(-\Delta)^{-1/2}$ ($j = 1, 2, \dots, n$) on the Q-type spaces $Q_\alpha^\beta(\mathbb{R}^n)$. As an application, we get the well-posedness and regularity of the quasi-geostrophic equation with initial data in $Q_\alpha^{\beta,-1}(\mathbb{R}^2)$.

1. INTRODUCTION

In this paper, we consider the boundedness of Riesz transforms on the space $Q_\alpha^\beta(\mathbb{R}^n)$, which was introduced in [18] and defined as the set of all measurable functions with

$$\sup_I (l(I))^{2\alpha-n+2\beta-2} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha-2\beta+2}} dx dy < \infty$$

with the supremum being taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n . For $\beta = 1$, the space $Q_\alpha^\beta(\mathbb{R}^n)$ becomes the classical space $Q_\alpha(\mathbb{R}^n)$ defined by the following norm:

$$(1.1) \quad \|f\|_{Q_\alpha} = \sup_I \left((l(I))^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dx dy \right)^{1/2} < \infty.$$

This space was first introduced by M. Essén, S. Janson, L. Peng and J. Xiao in [10]. As a new space between Sobolev spaces $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, it has been studied extensively by many authors. We refer the readers to [10], [29] and [9] for further information and details.

Since the space $Q_\alpha(\mathbb{R}^n)$ own a structure similar to $BMO(\mathbb{R}^n)$, it can be regarded as an analogy of $BMO(\mathbb{R}^n)$ in many cases. It is well-known that by the equivalent characterization of Hardy space $H^1(\mathbb{R}^n)$, Riesz transforms $R_j = \partial_j(-\Delta)^{-1/2}$, $j = 1, 2, \dots, n$ are bounded on $H^1(\mathbb{R}^n)$. Then the duality between $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ obviously implies the boundedness of $R_j = \partial_j(-\Delta)^{-1/2}$ on $BMO(\mathbb{R}^n)$. So it is natural to ask if $R_j, j = 1, 2, \dots, n$ are bounded on $Q_\alpha^\beta(\mathbb{R}^n)$. In Section 2, by an equivalent characterization of $Q_\alpha^\beta(\mathbb{R}^n)$ associated with fractional heat semi-group $e^{-t(-\Delta)^\beta}$ obtained in [18], we prove that Riesz transforms R_j are bounded on the space $Q_\alpha^\beta(\mathbb{R}^n)$. As far as we know, our result is new even in the case $Q_\alpha(\mathbb{R}^n), \alpha \in (0, 1)$.

As an application, we consider the well-posedness and regularity of the quasi-geostrophic equation with initial data in $Q_\alpha^{\beta,-1}(\mathbb{R}^n)$. In recent years, Q-type spaces

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have been applied to the study of PDE and Harmonic analysis by several authors. For example, in [29], J. Xiao replaced $BMO^{-1}(\mathbb{R}^n)$ in [15] by a new critical space $Q_\alpha^{-1}(\mathbb{R}^n)$ which is derivatives of Q_α , $\alpha \in (0, 1)$ and got the well-posedness of Navier-Stokes equations with initial data in $Q_\alpha^{-1}(\mathbb{R}^n)$. When $\alpha = 0$, $Q_\alpha^{-1}(\mathbb{R}^n) = BMO^{-1}(\mathbb{R}^n)$, his result generalized the well-posedness obtained by Koch and Tataru in [15].

In [18], inspiring by [29] and the scaling invariant, we introduced a new Q-type space $Q_\alpha^\beta(\mathbb{R}^n)$ with $\alpha > 0$, $\max\{\frac{1}{2}, \alpha\} < \beta < 1$ such that $\alpha + \beta - 1 \geq 0$ and considered the generalized Navier-Stokes equations as follows.

$$(1.2) \quad \begin{cases} \partial_t u + (-\Delta)^\beta u + (u \cdot \nabla)u - \nabla p = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}_+^{1+n}; \\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^n. \end{cases}$$

We proved the well-posedness and regularity of the generalized Navier-Stokes equations with initial data in the space $Q_\alpha^{\beta, -1}$. For $\beta = 1$, our spaces $Q_\alpha^{\beta, -1}$ retreat to Q_α^{-1} in [29]. So our result can be regarded as a generalization of those of [15] and [29].

In Section 3, We consider the two-dimensional subcritical quasi-geostrophic dissipative equation $(DQG)_\beta$.

$$(1.3) \quad \begin{cases} \partial_t \theta + (-\Delta)^\beta u + (u \cdot \nabla)\theta = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \alpha > 0; \\ u = \nabla^\perp (-\Delta)^{-1/2} \theta; \\ \theta(0, x) = \theta_0, & \text{in } \mathbb{R}^2. \end{cases}$$

where $\beta \in (\frac{1}{2}, 1)$, the scalar θ represent the potential temperature, and u is the fluid velocity.

The equations $(DQG)_\beta$ in either inviscid or dissipative form, are special cases of the general quasi-geostrophic approximations for the atmosphere and ocean flow with small Rossby and Ekman numbers. Therefore, they are important models in geophysical fluid dynamics. It was proposed by P. Constantin and A. Majda, etc that the equations $(DQG)_\beta$ can be regarded as low dimensional model equations for mathematical study of possible development of singularity in smooth solutions of unforced incompressible three dimensional fluid equations. See e.g. [7], [11], [12], [22], [23] and the references therein.

Recently, the equations $(DQG)_\beta$ have been intensively studied because of their importance in mathematical and geophysical fluid dynamics as mentioned above. Some important progress has been made. We refer the readers to [2], [3], [4], [5], [6], [8], [13], [27], [28] etc. for details.

In [20], F. Marchand and P. G. Lemarié-Rieusset studied the equations $(DQG)_\beta$ and get the well-posedness of the solutions to the equation $(DQG)_1$ with the initial data in $BMO^{-1}(\mathbb{R}^2)$. However, because the space $BMO^{-1}(\mathbb{R}^2)$ is invariant under the scaling: $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$, we see that for the scaling corresponding to general $\beta < 1$,

$$(1.4) \quad \theta_\lambda(t, x) = \lambda^{2\beta-1} \theta(\lambda^{2\beta} t, \lambda x), \quad \theta_{0,\lambda}(x) = \lambda^{2\beta-1} \theta_0(\lambda x),$$

the space $BMO^{-1}(\mathbb{R}^2)$ is not invariant under this scaling.

The above observation implies that if we want to generalize the result in [20] to general $\beta < 1$, we should choose a new space X^β which satisfies the following two properties. At first, the space X^β should be invariant under the scaling (1.4).

Secondly, $BMO^{-1}(\mathbb{R}^2)$ is a “special” case of X^β , that is, when $\beta = 1$, $X^\beta = BMO^{-1}(\mathbb{R}^2)$.

In [18], we have proved the space $Q_\alpha^{\beta, -1}(\mathbb{R}^2)$ is exactly such a space. Therefore we could apply our approach in [18] to the $(DQG)_\beta$ equation and get the well-posedness and regularity of the solution to the $(DQG)_\beta$ equation.

It should be pointed out that the scope of β is refined by the choice of the space $Q_\alpha^\beta(\mathbb{R}^n)$. In the definition of $Q_\alpha^\beta(\mathbb{R}^n)$, the parameters $\{\alpha, \beta\}$ should satisfy the condition: $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ and $\alpha < \beta$ with $\alpha + \beta - 1 \geq 0$ (see [18]). It is easy to see that $\beta > \frac{1}{2}$.

In [24], the authors proved the global existence of the solutions of the subcritical quasi-geostrophic equations with small size initial data in the Besov norms spaces $B_{\infty}^{1-2\beta, \infty}(\mathbb{R}^2)$. However our well-posedness can't be deduced by the existence result in [24]. In addition, owing to the structure of the Q_α^β , we can apply the method in [18] to get the regularity of the solutions to the equation $(DQG)_\beta$.

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2. RIESZ TRANSFORM ON Q-TYPE SPACES $Q_\alpha^\beta(\mathbb{R}^n)$

In this section, we will prove that Riesz transforms are bounded on Q-type spaces $Q_\alpha^\beta(\mathbb{R}^n)$. At first we recall the definition of $Q_\alpha^\beta(\mathbb{R}^n)$.

Definition 2.1. Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if

$$\sup_I (l(I))^{2\alpha-n+2\beta-2} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha-2\beta+2}} dx dy < \infty$$

where the supremum is taken over all cubes I with the edge length $l(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

For $\beta = 1$ and $\alpha > -\infty$, the above spaces become the $Q_\alpha(\mathbb{R}^n)$ which were introduced in [10] by M. Essen, S. Janson, L. Peng and J. Xiao. In [9], G. Dafni and J. Xiao further studied the structure of this space and get an equivalent characterization via the heat semigroup associated with Δ . It has been proved that when $\alpha \in (0, 1)$, $Q_\alpha(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$ and when $-\infty < \alpha < 0$, $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ (See [10]). Recall the definition of Morrey space $\mathcal{L}_{p, \lambda}(\mathbb{R}^n)$:

$$(2.1) \quad \|f\|_{\mathcal{L}_{p, \lambda}} = \sup_I \left((l(I))^{-\lambda} \int_I |f(x) - f_I|^p dx \right)^{1/p} < \infty.$$

We see that when $\lambda = n$, $\mathcal{L}_{p, \lambda}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ by John-Nirenberg inequality. It is natural to ask if there exists some relation between $\mathcal{L}_{p, \lambda}(\mathbb{R}^n)$ and $Q_\alpha(\mathbb{R}^n)$. In fact, in [29], J. Xiao proved that for $\alpha \in (0, 1)$, $Q_\alpha(\mathbb{R}^n) = (-\Delta)^{-\frac{\alpha}{2}} \mathcal{L}_{2, n-2\alpha}(\mathbb{R}^n)$.

Following Xiao's idea in [29], we will prove for our space $Q_\alpha^\beta(\mathbb{R}^n)$, a similar result holds. At first we proved an equivalent characterization of $\mathcal{L}_{2, n-2\gamma}(\mathbb{R}^n)$ via the semigroup $e^{-t(-\Delta)^\beta}$. Here $e^{-t(-\Delta)^\beta}$ denotes the convolution operator defined by Fourier transform:

$$\widehat{e^{-t(-\Delta)^\beta} f}(\xi) = e^{-t|\xi|^{2\beta}} \widehat{f}(\xi).$$

Lemma 2.2. *For $\gamma \in (0, 1)$. Let f be a measurable complex-valued function on \mathbb{R}^n . Then*

$$f \in \mathcal{L}_{2, n-2\gamma}(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\gamma-n} \int_0^r \int_{|y-x| < r} \left| \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y) \right|^2 t dy dt < \infty.$$

Proof. Taking $(\psi_0)_t(x) = t \nabla e^{-t^{2\beta}(-\Delta)^\beta}(x, 0)$ with the Fourier symbol $\widehat{(\psi_0)_t(x)}(\xi) = t|\xi|e^{-t^{2\beta}|\xi|^{2\beta}}$. Define a ball $B = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $f_{2B} = \frac{1}{|2B|} \int_{2B} f(x) dx$ is the mean of f on $2B$. We split f into $f = f_1 + f_2 + f_3$ where $f_1 = (f - f_{2B})\chi_{2B}$, $f_2 = (f - f_{2B})\chi_{(2B)^c}$ and $f_3 = f_{2B}$. Because

$$\int (\psi_0)_t(x) dx = \int t \nabla e^{-t^{2\beta}(-\Delta)^\beta}(x, 0) dx = 0,$$

we have

$$t \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y) = (\psi_0)_t * f(y) = (\psi_0)_t * f_1(y) + (\psi_0)_t * f_2(y).$$

It is easy to see that

$$\begin{aligned} \int_0^r \int_B |(\psi_0)_t * f_1(y)|^2 \frac{dy dt}{t} &\lesssim \int_0^r \int_{\mathbb{R}^n} |(\psi_0)_t * f_1(y)|^2 \frac{dy dt}{t} \\ &= \left\| \left(\int_0^\infty |(\psi_0)_t * f_1(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2(dy)}. \end{aligned}$$

Because $(\psi_0)_1 = \nabla e^{-(\Delta)^\beta}$, obviously we have $\int (\psi_0)_1(x) dx = 1$ and $(\psi_0)_1$ belongs to the Schwartz class \mathcal{S} , the function

$$G(f) = \left(\int_0^\infty |(\psi_0)_t * f_1(y)|^2 \frac{dt}{t} \right)^{1/2}$$

is a Littlewood-Paley G-function. So we can get

$$\begin{aligned} \int_0^r \int_B |(\psi_0)_t * f_1(y)|^2 \frac{dy dt}{t} &\lesssim \int_{2B} |f(y) - f_{2B}|^2 dy \\ &\lesssim r^{n-2\gamma} \|f\|_{\mathcal{L}_{2, n-2\gamma}(\mathbb{R}^n)}^2. \end{aligned}$$

Now we estimate the term associated with $f_2(y)$. Because

$$\begin{aligned} |(\psi_0)_t * f_2(y)| &= \left| \int_{\mathbb{R}^n} t \nabla e^{-t^{2\beta}(-\Delta)^\beta}(y-z) f_2(z) dz \right| \\ &\lesssim \int_{\mathbb{R}^n \setminus 2B} |t \nabla e^{-t^{2\beta}(-\Delta)^\beta}(y-z)| |f(z) - f_{2B}| dz \\ &\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{t |f(z) - f_{2B}|}{t^{n+1} (1 + t^{-1}|z-y|)^{n+1}} dz \end{aligned}$$

where in the last inequality, we have used the following estimate:

$$\left| \nabla e^{-t^{2\beta}(-\Delta)^\beta}(x, y) \right| \lesssim \frac{1}{t^{\frac{n+1}{2\beta}}} \frac{1}{(1 + t^{-\frac{1}{2\beta}}|x-y|)^{n+1}}.$$

Set $B_k = B(x, 2^k)$. For every $(t, y) \in (0, r) \times B(x, r)$, we have $0 < t < r$ and $|x - y| < r$. If $z \in B_{k+1} \setminus B_k$, that is, $|z - x| > 2^k r$, we have $|x - y| < |x - z|/2$ and

$$\begin{aligned}
|(\psi_0)_t * f_2(y)| &\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{t|f(z) - f_{2B}|}{(t + |z - x|)^{n+1}} dz \\
&\lesssim \sum_{k=1}^{\infty} \frac{t}{(2^k r)^{n+1}} \int_{2^{k+1}B} |f(z) - f_{2B}| dz \\
&\lesssim t \sum_{k=1}^{\infty} \frac{(2^{k+1}r)^n}{(2^k r)^{n+1}} \left(\frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} |f(z) - f_{2B}|^2 dz \right)^{1/2} \\
&\lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} \left[\left(\frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} |f(z) - f_{2^{k+1}B}|^2 dz \right)^{1/2} + |f_{2^{k+1}B} - f_{2B}| \right] \\
&\lesssim t \left[\sum_{k=1}^{\infty} \frac{1}{2^k r} \left(\frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} |f(z) - f_{2^{k+1}B}|^2 dz \right)^{1/2} + \sum_{k=1}^{\infty} \frac{1}{2^k r} |f_{2^{k+1}B} - f_{2B}| \right] \\
&=: t(S_1 + S_2).
\end{aligned}$$

For S_1 , we have

$$\begin{aligned}
S_1 &= t \sum_{k=1}^{\infty} \frac{1}{2^k r} \left(\frac{(2^{k+1}r)^{n-2\gamma}}{(2^{k+1}r)^n} \frac{1}{(2^{k+1}r)^{n-2\gamma}} \int_{2^{k+1}B} |f(z) - f_{2^{k+1}B}|^2 dz \right)^{1/2} \\
&\lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} r^{-\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)} \\
&\lesssim tr^{-1-\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}.
\end{aligned}$$

For S_2 , we have

$$S_2 \lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} [|f_{2B} - f_{4B}| + \cdots + |f_{2^k B} - f_{2^{k+1}B}|].$$

For $\forall 2 \leq j \leq k$, it is easy to see that

$$\begin{aligned}
|f_{2^j B} - f_{2^{j+1}B}| &\lesssim \frac{1}{|2^j B|} \int_{2^j B} |f(z) - f_{2^{j+1}B}| dz \\
&\lesssim \left(\frac{1}{|2^j B|} \int_{2^j B} |f(z) - f_{2^{j+1}B}|^2 dz \right)^{1/2} \\
&\lesssim r^{-\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}.
\end{aligned}$$

Then we can get

$$S_2 \lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} k \cdot r^{-\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)} \lesssim tr^{-1-\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}.$$

Therefore we can get

$$\begin{aligned}
\int_0^r \int_B |(\psi_0)_t * f_2(y)|^2 t^{-1} dy dt &\lesssim \int_0^r \int_B t^2 r^{-2\gamma-2} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}^2 dy dt \\
&\lesssim \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}^2 r^{-2\gamma-2} |B| \int_0^r t dt \\
&\lesssim r^{n-2\gamma} \|f\|_{\mathcal{L}_{2,n-2\gamma}(\mathbb{R}^n)}^2.
\end{aligned}$$

For the converse, let $S(I) = \{(t, x) \in \mathbb{R}_+^{n+1}, 0 < t < l(I), x \in I\}$ if f such that

$$\begin{aligned} & \sup_I [l(I)]^{2\gamma-n} \int_{S(I)} \left| t \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y) \right|^2 \frac{dydt}{t} \\ &= \sup_I [l(I)]^{2\gamma-n} \int_{S(I)} \left| \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y) \right|^2 t dydt < \infty. \end{aligned}$$

Denote

$$\Pi_{\psi_0} F(x) = \int_{\mathbb{R}_+^{n+1}} F(t, y) (\psi_0)_t(x - y) \frac{dydt}{t},$$

we will prove that if

$$\|F\|_{C_\gamma} = \sup_I \left([l(I)]^{2\gamma-n} \int_{S(I)} |F(t, y)|^2 \frac{dydt}{t} \right)^{1/2} < \infty,$$

then for any cube $J \subset \mathbb{R}^n$

$$\int_J |\Pi_{\psi_0} F(x) - (\Pi_{\psi_0} F)_J|^2 dx \lesssim [l(J)]^{n-2\gamma} \|F\|_{C_\gamma}^2.$$

We split F into $F = F_1 + F_2 = F|_{S(2J)} + F|_{\mathbb{R}^{n+1} \setminus S(2J)}$ and get

$$\begin{aligned} \int_J |\Pi_{\psi_0} F_1(x)|^2 dx &\leq \int_J |\Pi_{\psi_0} F_1(x)|^2 dx \\ &\leq \int_{S(2J)} |F(t, y)|^2 \frac{dydt}{t} \\ &\lesssim [l(J)]^{n-2\gamma} \|F\|_{C_\gamma}^2. \end{aligned}$$

Now we estimate the term associated with F_2 . We have

$$\begin{aligned} & \int_J |\Pi_{\psi_0} F_1(x)|^2 dx \\ &= \int_J \left| \int_{\mathbb{R}_+^{n+1}} (\psi_0)_t(x - y) F_2(t, y) t^{-1} dydt \right|^2 dx \\ &\lesssim \int_J \left(\int_{\mathbb{R}_+^{n+1} \setminus S(2J)} |(\psi_0)_t(x - y)| |F_2(t, y)| \frac{dydt}{t} \right)^2 dx \\ &= \int_J \left(\sum_{k=1}^{\infty} \int_{S(2^{k+1}J) \setminus S(2^k J)} |(\psi_0)_t(x - y)| |F_2(t, y)| \frac{dydt}{t} \right)^2 dx. \end{aligned}$$

Because

$$|(\psi_0)_t(x - y)| \lesssim \frac{t}{t^{n+1}(1 + t^{-1}|x - y|)^{n+1}},$$

we have

$$\begin{aligned}
& \int_J |\Pi_{\psi_0} F_1(x)|^2 dx \\
& \lesssim \int_J \left(\sum_{k=1}^{\infty} \int_{S(2^{k+1}J) \setminus S(2^k J)} \frac{t}{[t + 2^k l(J)]^{n+1}} |F_2(t, y)| \frac{dydt}{t} \right)^2 dx \\
& \lesssim \int_J \left(\sum_{k=1}^{\infty} (2^k l(J))^{-(n+1)} \int_{S(2^{k+1}J) \setminus S(2^k J)} |F_2(t, y)| dydt \right)^2 dx \\
& \lesssim \int_J \left[\sum_{k=1}^{\infty} [2^k l(J)]^{-n} [2^{k+1} l(J)]^{n/2} \left(\int_{S(2^{k+1}J) \setminus S(2^k J)} |F_2(t, y)|^2 \frac{dydt}{t} \right)^{1/2} \right]^2 dx \\
& \lesssim \|F\|_{C_\gamma}^2 [l(J)]^{n-2\gamma}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\int_J |\Pi_{\psi_0} F(x) - (\Pi_{\psi_0} F)_J|^2 dx & \leq \int_J |\Pi_{\psi_0} F(x)|^2 dx \\
& \lesssim \int_J |\Pi_{\psi_0} F_1(x)|^2 dx + \int_J |\Pi_{\psi_0} F_2(x)|^2 dx \\
& \lesssim \|F\|_{C_\gamma}^2 [l(J)]^{n-2\gamma}.
\end{aligned}$$

Because

$$\Pi_{\psi_0} F(x) = \int (\psi_0)_t * (\psi_0)_t * f \frac{dt}{t},$$

by Calderón reproducing formula, we have $\Pi_{\psi_0} F(x) = f(x)$, that is, $f(x) = \Pi_{\psi_0} F(x) \in \mathcal{L}_{2, n-2\gamma}(\mathbb{R}^n)$. This completes the proof of Lemma 2.2. \square

Theorem 2.3. For $\alpha > 0$, $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we have

$$Q_\alpha^\beta(\mathbb{R}^n) = (-\Delta)^{-\frac{(\alpha-\beta+1)}{2}} \mathcal{L}_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n).$$

Proof. For $f \in \mathcal{L}_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n)$. Let $F(t, y) = t^{\alpha-\beta+1} t \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)$. By Lemma 2.2, we have

$$\begin{aligned}
& r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |F(t, y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \\
& \lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t^{\alpha-\beta+1} t \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \\
& \lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t \nabla e^{-t^{2\beta}(-\Delta)^\beta} f(y)|^2 \frac{dydt}{t} \\
& \lesssim \|f\|_{\mathcal{L}_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n)}.
\end{aligned}$$

So $F \in T_{\alpha, \beta}^\infty$ where the space $T_{\alpha, \beta}^\infty$ is a tent space defined in [18] (See [18], Definition 3.5 for details). By Theorem 3.21 in [18], Π_{ψ_0} is bounded from $T_{\alpha, \beta}^\infty$ to Q_α^β . Therefore we have

$$\|f\|_{Q_\alpha^\beta(\mathbb{R}^n)} = \|\Pi_{\psi_0} F\|_{Q_\alpha^\beta(\mathbb{R}^n)} \lesssim \|F\|_{T_{\alpha, \beta}^\infty}.$$

We have

$$\widehat{F}(t, \xi) = t^{\alpha-\beta+2} |\xi| e^{-t^{2\beta} |\xi|^{2\beta}} \widehat{f}(\xi),$$

then

$$\begin{aligned}
\widehat{\Pi_{\psi_0} F}(\xi) &= \int_0^\infty \widehat{F}(t, \xi) \widehat{(\psi_0)_t}(\xi) \frac{dt}{t} \\
&= \int_0^\infty t^{\alpha-\beta+2} |\xi| e^{-t^{2\beta} |\xi|^{2\beta}} t |\xi| e^{-t^{2\beta} |\xi|^{2\beta}} \widehat{f}(\xi) \frac{dt}{t} \\
&= |\xi|^2 \widehat{f}(\xi) \int_0^\infty t^{\alpha-\beta+2} e^{-t^{2\beta} |\xi|^{2\beta}} dt.
\end{aligned}$$

Set $t^{2\beta} = s$ and $|\xi|^{2\beta} s = u$. We can get

$$\begin{aligned}
\widehat{\Pi_{\psi_0} F}(\xi) &= \int_0^\infty s^{\frac{\alpha-\beta+2}{2\beta}} e^{-2s |\xi|^{2\beta}} s^{\frac{1}{2\beta}-1} ds \widehat{f}(\xi) |\xi|^2 \\
&= \widehat{f}(\xi) |\xi|^2 \int_0^\infty (u |\xi|^{-2\beta})^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-u} |\xi|^{-2\beta} du \\
&= \widehat{f}(\xi) |\xi|^2 |\xi|^{-(\alpha-\beta+3)+2\beta-2\beta} \int_0^\infty u^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-2u} du.
\end{aligned}$$

Because $\frac{1}{2} < \beta < 1$ and $0 < \alpha < \beta$, the integral $\int_0^\infty u^{\frac{\alpha-\beta+3}{2\beta}-1} e^{-2u} du$ is a constant. We denote it by $C_{\alpha, \beta}$, so

$$\widehat{\Pi_{\psi_0} F}(\xi) = C_{\alpha, \beta} \widehat{f}(\xi) |\xi|^{-(\alpha-\beta+1)}.$$

By inverse Fourier transform, we have

$$\Pi_{\psi_0} F(x) = C_{\alpha, \beta} (-\Delta)^{-\frac{\alpha-\beta+1}{2}} f(x).$$

Conversely, suppose $g \in Q_\alpha^\beta(\mathbb{R}^n)$. Set $G(t, y) = t^{1-(\alpha-\beta+1)} \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y)$. We have, by the equivalent characterization of $Q_\alpha^\beta(\mathbb{R}^n)$ (see [18] for details).

$$\begin{aligned}
&\left([l(I)]^{2(\alpha+\beta-1)-n} \int_{S(I)} \left| t^{1-2(\alpha-\beta+1)} \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y) \right|^2 \frac{dy dt}{t} \right)^{1/2} \\
&= \left([l(I)]^{2(\alpha+\beta-1)-n} \int_{S(I)} \left| t \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y) \right|^2 \frac{dy dt}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2} \\
&\lesssim \|g\|_{Q_\alpha^\beta(\mathbb{R}^n)},
\end{aligned}$$

that is, $G(t, y) \in C_{\alpha+\beta-1}$. By Lemma 2.2, we have $\Pi_{\psi_0} G(t, y) \in \mathcal{L}_{2, n-2(\alpha+\beta-1)}$. We have

$$\begin{aligned}
\widehat{f}(\xi) &= \widehat{\Pi_{\psi_0} G}(t, \xi) \\
&= \int_0^\infty t |\xi| e^{-t^{2\beta} |\xi|^{2\beta}} t^{1-(\alpha-\beta+1)} |\xi| e^{-t^{2\beta} |\xi|^{2\beta}} \widehat{g}(\xi) \frac{dt}{t} \\
&= C_{\alpha, \beta} |\xi|^{1+(\alpha-\beta)} \widehat{g}(\xi) \\
&= C_{\alpha, \beta} [(-\Delta)^{\frac{\alpha-\beta+1}{2}} g](\xi).
\end{aligned}$$

Then $f(x) = C_{\alpha, \beta} (-\Delta)^{\frac{\alpha-\beta+1}{2}} g$. Thus, we get $Q_\alpha^\beta(\mathbb{R}^n) = (-\Delta)^{-\frac{(\alpha-\beta+1)}{2}} \mathcal{L}_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n)$. \square

Theorem 2.4. Suppose $\alpha > 0$, $\max \alpha, \frac{1}{2} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. For $j = 1, 2, \dots, n$, the Riesz transforms $R_j = \partial_j (-\Delta)^{-1/2}$ are bounded on the Q -type spaces $Q_\alpha^\beta(\mathbb{R}^n)$.

Proof. Notice the equivalent norm of $Q_\alpha^\beta(\mathbb{R}^n)$. $f \in Q_\alpha^\beta(\mathbb{R}^n)$ if and only if

$$(2.2) \quad \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} |\nabla e^{-t(-\Delta)^\beta} f(y)|^2 t^{-\frac{\alpha}{\beta}} dy dt < \infty.$$

As a convolution operator, Riesz transform R_j and ∇ can change the order of operation. So we only need to estimate the term

$$r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |R_j f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}}$$

where $0 < \alpha < \beta$, $\alpha + \beta - 1 \geq 0$ and $j = 1, 2, \dots, n$. We split $f(t, y)$ into

$$f(t, y) = f_0(t, y) + \sum_{k=1}^{\infty} f_k(t, y),$$

where $f_0(t, y) = f(t, y)\chi_{B(x_0, 2r)}(y)$ and $f_k(t, y) = f(t, y)\chi_{B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)}(y)$. We have

$$\begin{aligned} & \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |R_j f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \\ & \leq \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |R_j f_0(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \\ & \quad + \sum_{k=1}^{\infty} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |R_j f_k(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \\ & := M_0 + \sum_{k=1}^{\infty} M_k. \end{aligned}$$

By the L^2 -boundedness of Riesz transforms R_j , $j = 1, 2, \dots, n$, we have

$$\begin{aligned} M_0 & \lesssim \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \\ & \lesssim C_\alpha \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \end{aligned}$$

Now we estimate M_k . We only need to estimate the integration as follows.

$$I = \int_{|y-x_0| < r} |R_j f_k(t, y)|^2 dy.$$

As a singular integral operator,

$$R_j g(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} g(y) dy.$$

By Hölder's inequality, we can get

$$\begin{aligned}
I &= \int_{|y-x_0|<r} \left| \int_{2^k r \leq |z-x_0|<2^{k+1}r} \frac{y_j - z_j}{|y-z|^{n+1}} f(t, z) dz \right|^2 dy \\
&\lesssim \int_{|y-x_0|<r} \left(\frac{1}{(2^k r)^n} \int_{|z-x_0|<2^{k+1}r} |f(t, z)| dz \right)^2 dy \\
&\lesssim \int_{|y-x_0|<r} \frac{1}{(2^k r)^n} \int_{|z-x_0|<2^{k+1}r} |f(t, z)|^2 dz dy \\
&\lesssim \frac{1}{2^{kn}} \int_{|z-x_0|<2^{k+1}r} |f(t, z)|^2 dz.
\end{aligned}$$

So we have

$$\begin{aligned}
M_k &= \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2} \\
&\lesssim \left(r^{2\alpha-n+2\beta-2} \frac{1}{2^{kn}} \int_0^{r^{2\beta}} \int_{|z-x_0|<2^{k+1}r} |f(t, z)|^2 \frac{dz dt}{t^{\alpha/\beta}} \right)^{1/2} \\
&\lesssim \left(2^{-k(2\alpha-n+2\beta-2)} \frac{1}{2^{kn}} (2^k r)^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|z-x_0|<2^{k+1}r} |f(t, z)|^2 \frac{dz dt}{t^{\alpha/\beta}} \right)^{1/2} \\
&\lesssim (2^{-k(2\alpha-n+2\beta-2)} \frac{1}{2^{kn}})^{1/2} \sup_{x_0 \in \mathbb{R}^n, r>0} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|z-x_0|<r} |f(t, z)|^2 \frac{dz dt}{t^{\alpha/\beta}} \right)^{1/2}.
\end{aligned}$$

Therefore we can get

$$\begin{aligned}
M_0 + \sum_{k=1}^{\infty} M_k &\lesssim [1 + \sum_{k=1}^{\infty} 2^{-k(\alpha+\beta-1)}] \sup_{x_0 \in \mathbb{R}^n, r>0} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|z-x_0|<r} |f(t, z)|^2 \frac{dz dt}{t^{\alpha/\beta}} \right)^{1/2} \\
&\lesssim C \|f\|_{Q_{\alpha}^{\beta}(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of Theorem 2.4. \square

Similar to the proof of Theorem 2.4, we can prove the following theorem.

Theorem 2.5. *For a singular operator T defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

where the kernel $K(x)$ satisfies:

$$|\partial_x^{\gamma} K(x)| \leq A_{\gamma} |x|^{-n-\gamma}, \quad (\gamma > 0)$$

or equivalently $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$ with the symbol $m(\xi)$ satisfies:

$$|\partial_{\xi}^{\gamma} m(\xi)| \leq A_{\gamma'} |\xi|^{-\gamma}$$

holds for all γ . Suppose $\alpha > 0$, $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we have T is bounded on the Q -type spaces $Q_\alpha^\beta(\mathbb{R}^n)$.

3. WELL-POSEDNESS AND REGULARITY OF QUASI-GEOSTROPHIC EQUATION

In this section, we study the well-posedness and regularity of quasi-geostrophic equation with initial data in the space $Q_\alpha^\beta(\mathbb{R}^2)$. We introduce the definition of $X_\alpha^\beta(\mathbb{R}^n)$.

Definition 3.1. The space $X_\alpha^\beta(\mathbb{R}^2)$ consists of the functions which are locally integrable on $(0, \infty) \times \mathbb{R}^2$ such that

$$\sup_{t>0} t^{1-\frac{1}{2\beta}} \|f(t, \cdot)\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^2)} < \infty$$

and

$$\sup_{x \in \mathbb{R}^2, r>0} r^{2\alpha+2\beta-4} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |f(t, y)|^2 + |R_1 f(t, y)|^2 + |R_2 f(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} < \infty,$$

where R_j , ($j = 1, 2$) denote the Riesz transforms in \mathbb{R}^2 .

For the quasi-geostrophic dissipative equations

$$(3.1) \quad \begin{cases} \partial_t \theta = -(-\Delta)^\beta + \partial_1(\theta R_2 \theta) - \partial_2(\theta R_1 \theta); \\ \theta(0, x) = \theta_0(x) \end{cases}$$

where $\beta \in (\frac{1}{2}, 1)$. The solution to the equation (3.1) can be represented as

$$u(t, x) = e^{-t(-\Delta)^\beta} u_0 + B(u, u)$$

where the bilinear form $B(u, v)$ is defined by

$$B(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} (\partial_1(v R_2 u) - \partial_2(v R_1 u)) ds.$$

In order to prove the well-posedness, we need the following preliminary lemmas. For their proof, we refer the readers to [18], Lemma 4.8 and Lemma 4.9.

Lemma 3.2. Given $\alpha \in (0, 1)$. For a fixed $T \in (0, \infty]$ and a function $f(\cdot, \cdot)$ on \mathbb{R}_+^{1+n} , let $A(t) = \int_0^t e^{-(t-s)(-\Delta)^\beta} (-\Delta)^\beta f(s, x) ds$. Then

$$(3.2) \quad \int_0^T \|A(t, \cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^T \|f(t, \cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}.$$

Lemma 3.3. For $\beta \in (1/2, 1)$ and $N(t, x)$ defined on $(0, 1) \times \mathbb{R}^n$, let $A(N)$ be the quantity

$$A(\alpha, \beta, N) = \sup_{x \in \mathbb{R}^n, r \in (0, 1)} r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x|<r} |f(t, x)| \frac{dxdt}{t^{\alpha/\beta}}.$$

Then for each $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ there exists a constant $b(k)$ such that the following inequality holds:

$$(3.3) \quad \int_0^1 \left\| t^{\frac{k}{2}} (-\Delta)^{\frac{k\beta+1}{2}} e^{-\frac{t}{2}(-\Delta)^\beta} \int_0^t N(s, \cdot) ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \leq b(k) A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s, x)| \frac{dx ds}{s^{\alpha/\beta}}.$$

Remark 3.4. Similarly when $k = 0$, we can prove the following inequality:

$$(3.4) \quad \int_0^1 \left\| (-\Delta)^{\frac{1}{2}} e^{-t(-\Delta)^\beta} \int_0^t N(s, \cdot) ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s, x)| \frac{dx ds}{s^{\alpha/\beta}}.$$

Now we give the main result of this paper.

Theorem 3.5. (*Well-posedness*)

(i) *The subcritical quasi-geostrophic equation (3.1) has a unique small global mild solution in $(X_\alpha^\beta(\mathbb{R}^2))^2$ for all initial data θ_0 with $\nabla \cdot \theta = 0$ and $\|u_0\|_{Q_{\alpha,-1}^{\beta,-1}}$ being small.*

(ii) *For any $T \in (0, \infty)$, there is an $\varepsilon > 0$ such that the quasi-geostrophic equation (3.1) has a unique small mild solution in $(X_\alpha^\beta(\mathbb{R}^2))^2$ on $(0, T) \times \mathbb{R}^2$ when the initial data u_0 satisfies $\nabla \cdot u_0 = 0$ and $\|u_0\|_{(Q_{\alpha,-1}^{\beta,-1})^2} \leq \varepsilon$. In particular, for all $u_0 \in \overline{(VQ_{\alpha,-1}^{\beta,-1})^2}$ with $\nabla \cdot u_0 = 0$, there exists a unique small local mild solution in $(X_{\alpha,T}^\beta)^2$ on $(0, T) \times \mathbb{R}^2$.*

Proof. By the Picard contraction principle we only need to prove the bilinear form $B(u, v)$ is bounded on X_α^β . We split the proof into two parts.

Part I: $\dot{B}_\infty^{0,1}(\mathbb{R}^2)$ –boundedness. The proof of this part has been given in [20]. For completeness, we give the details. We have

$$\begin{aligned} & \|B(u, v)\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} \\ & \lesssim \int_0^t \|e^{-(t-s)(-\Delta)^\beta} (\partial_1(gR_2f) - \partial_2(gR_1f))\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} ds \\ & \lesssim \int_0^t \frac{C_\beta}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}} s^{1-\frac{1}{2\beta}} \|u\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} s^{1-\frac{1}{2\beta}} \|v\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} ds \\ & \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta} \int_0^t \frac{ds}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}}. \end{aligned}$$

Because when $\frac{1}{2} < \beta < 1$,

$$\int_0^{t/2} \frac{1}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}} ds \lesssim t^{\frac{1}{2\beta}-1}$$

and

$$\int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2\beta}} s^{1+(1-\frac{1}{\beta})}} ds \lesssim t^{-2+\frac{1}{\beta}} \int_{t/2}^t \frac{1}{(t-s)^{\frac{1}{2\beta}}} ds \lesssim t^{\frac{1}{2\beta}-1}.$$

Then we can get

$$t^{1-\frac{1}{2\beta}} \|B(u, v)\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}$$

where in the above estimates we have used for $f \in \dot{B}_{\infty}^{0,1}(\mathbb{R}^2)$, $\|R_j f\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^2)} \lesssim \|f\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^2)}$. In fact by Bernstein's inequality, we have

$$\begin{aligned} \sum_l \|\Delta_l R_j f\|_{L^{\infty}(\mathbb{R}^2)} &= \sum_l \|\partial_j (-\Delta)^{-1/2} \Delta_l f\|_{L^{\infty}(\mathbb{R}^2)} \\ &\lesssim \sum_l 2^l \|(-\Delta)^{-1/2} \Delta_l f\|_{L^{\infty}(\mathbb{R}^2)} \\ &\lesssim \sum_l 2^l 2^{-l} \|\Delta_l f\|_{L^{\infty}(\mathbb{R}^2)} \\ &\leq \|f\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^2)}. \end{aligned}$$

On the other hand, by Young's inequality, we have

$$t^{1-\frac{1}{2\beta}} \|e^{-t(-\Delta)^{\beta}} u_0\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^2)} \lesssim \|u_0\|_{\dot{B}_{\infty}^{1-2\beta,\infty}(\mathbb{R}^2)} \leq \|u_0\|_{Q_{\alpha}^{\beta,-1}(\mathbb{R}^2)}.$$

Part II: $L^2(\mathbb{R}^2)$ -boundedness. Now we estimate the operation of $B(u, v)$ on the Carleson part of X_{α}^{β} . We split again the estimate into two steps.

Step I: We want to prove the following estimate:

$$r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |B(u, v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_{\alpha}^{\beta}} \|v\|_{X_{\alpha}^{\beta}}.$$

By symmetry, we only need to deal with the term

$$\int_0^t e^{-(t-s)(-\Delta)^{\beta}} [\partial_1(v R_1 u)] ds = B_1(u, v) + B_2(u, v) + B_3(u, v)$$

where

$$B_1(u, v) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}} \partial_1[(1 - 1_{r,x})v R_1 u] ds,$$

$$B_2(u, v) = (-\Delta)^{-1/2} \partial_1 \int_0^t e^{-(t-s)(-\Delta)^{\beta}} (-\Delta)((-\Delta)^{1/2}(I - e^{-s(-\Delta)^{\beta}})(1_{r,x})v R_1 u) ds$$

and

$$B_3(u, v) = (-\Delta)^{-1/2} \partial_1 (-\Delta)^{1/2} e^{-t(-\Delta)^{\beta}} \int_0^t (1_{r,x})v R_1 u ds.$$

For B_1 . Because the n dimensional fractional heat kernel satisfies the following estimate:

$$(3.5) \quad |\nabla e^{-t(-\Delta)^{\beta}}(x, y)| \lesssim \frac{1}{t^{\frac{n+1}{2\beta}}} \frac{1}{\left(1 + \frac{|x-y|}{t^{1/2\beta}}\right)^{n+1}} \lesssim \frac{1}{(t^{\frac{1}{2\beta}} + |x-y|)^{n+1}},$$

we have, for $0 < t < r^{2\beta}$ and taking $n = 2$ in (3.5),

$$\begin{aligned} &|B_1(u, v)(t, x)| \\ &\lesssim \int_0^t \int_{|z-x| \geq 10r} \frac{|R_1 u(s, z)| |v(s, z)|}{|x-z|^{2+1}} dz ds \\ &\lesssim \left(\int_0^{r^{2\beta}} \int_{|z-x| \geq 10r} \frac{|R_1 u(s, z)|^2}{|x-z|^3} dz ds \right)^{1/2} \left(\int_0^{r^{2\beta}} \int_{|z-x| \geq 10r} \frac{|v(s, z)|^2}{|x-z|^3} dz ds \right)^{1/2} \\ &:= I_1 \times I_2. \end{aligned}$$

For I_1 , we have

$$\begin{aligned}
I_1 &\lesssim \left(\sum_{k=3}^{\infty} \frac{1}{(2^k r)^3} \int_0^{r^{2\beta}} \int_{|x-z| \leq 2^{k+1}r} |R_1 u(s, x)|^2 ds dx \right)^{1/2} \\
&\lesssim \left(\sum_{k=3}^{\infty} \frac{1}{(2^k r)^3} (2^k r)^{2\alpha+2\beta-2} (2^k r)^{2-2\beta} \int_0^{r^{2\beta}} \int_{|x-z| \leq 2^{k+1}r} |R_1 u(s, x)|^2 \frac{ds dx}{s^{\alpha/\beta}} \right)^{1/2} \\
&\lesssim \left(\sum_{k=3}^{\infty} \frac{(2^k r)^{2-2\beta}}{(2^k r)^3} (2^k r)^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-z| \leq 2^{k+1}r} |R_1 u(s, x)|^2 ds dx \right)^{1/2} \\
&\lesssim \|u\|_{X_\alpha^\beta} \left(\sum_{k=3}^{\infty} \frac{1}{2^{k(2\beta-1)}} \frac{1}{r^{2\beta-1}} \right)^{1/2} \\
&\lesssim \left(\frac{1}{r^{2\beta-1}} \right)^{1/2} \|u\|_{X_\alpha^\beta}.
\end{aligned}$$

Similarly we can get $I_2 \lesssim \left(\frac{1}{r^{2\beta-1}} \right)^{1/2} \|v\|_{X_\alpha^\beta}$ and therefore we have

$$|B_1(u, v)| \lesssim \frac{1}{r^{2\beta-1}} \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}.$$

Then we can get, using $0 < \alpha < \beta$,

$$\begin{aligned}
\int_0^{r^{2\beta}} \int_{|x-y| < r} |B_1(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} &\lesssim \frac{1}{r^{4\beta-2}} r^2 \int_0^{r^{2\beta}} \frac{dt}{t^{\alpha/\beta}} \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2 \\
&\lesssim \frac{1}{r^{4\beta-2}} r^2 r^{2\beta-2\alpha} \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2 \\
&\lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2.
\end{aligned}$$

That is to say

$$r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y| < r} |B_1(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2.$$

For B_2 . By the L^2 -boundedness of Riesz transform, we have

$$\begin{aligned}
&\int_0^{r^{2\beta}} \int_{|x-y| < r} |B_2(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^\beta} \left\| \int_0^t e^{-(t-s)(-\Delta)^\beta} (-\Delta) ((-\Delta)^{-1/2} (I - e^{-s(-\Delta)^\beta}) (1_{r,x}) v R_1 u) ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^\beta} \left\| \int_0^t e^{-(t-s)(-\Delta)^\beta} (-\Delta)^\beta ((-\Delta)^{1/2-\beta} (I - e^{-s(-\Delta)^\beta}) (1_{r,x}) v R_1 u) ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \\
&\lesssim \int_0^{r^{2\beta}} t^{2-\frac{1}{\beta}} \int_{|y-x| < r} |R_1 u(t, y)|^2 |v(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \\
&\lesssim \left(\sup_{t>0} t^{1-\frac{1}{2\beta}} \|R_1 u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \right) \left(\sup_{t>0} t^{1-\frac{1}{2\beta}} \|v(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \right) \\
&\times \int_0^{r^{2\beta}} \int_{|y-x| < r} |R_1 u(t, y)| |v(t, y)| \frac{dt dy}{t^{\alpha/\beta}}.
\end{aligned}$$

On one hand, by Bernstein's inequality, we have

$$\|R_1 u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \|R_1 u(t, \cdot)\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)} \lesssim \|u(t, \cdot)\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)}.$$

Then we get

$$\sup_{t>0} t^{1-\frac{1}{2\beta}} \|R_1 u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \lesssim \sup_{t>0} t^{1-\frac{1}{2\beta}} \|u(t, \cdot)\|_{\dot{B}_\infty^{0,1}(\mathbb{R}^2)}.$$

On the other hand, we have, by Hölder's inequality,

$$\begin{aligned} & \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_1 u(t, y)| |v(t, y)| \frac{dt dy}{t^{\alpha/\beta}} \\ & \lesssim \left(\int_0^{r^{2\beta}} \int_{|y-x|<r} |R_1 u(t, y)|^2 \frac{dt dy}{t^{\alpha/\beta}} \right)^{1/2} \left(\int_0^{r^{2\beta}} \int_{|y-x|<r} |v(t, y)|^2 \frac{dt dy}{t^{\alpha/\beta}} \right)^{1/2} \\ & \lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2. \end{aligned}$$

Hence we get

$$\int_0^{r^{2\beta}} \int_{|x-y|<r} |B_2(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^\beta}^2 \|v\|_{X_\alpha^\beta}^2.$$

For $B_3(u, v)$. We have

$$\begin{aligned} & \int_0^{r^{2\beta}} \int_{|y-x|<r} |B_3(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \\ & = \int_0^{r^{2\beta}} \int_{|y-x|<r} \left| (-\Delta)^{-1/2} \partial_1 (-\Delta)^{1/2} e^{-t(-\Delta)^\beta} \left(\int_0^t (1_{r,x}) v R_1 u dh \right) \right|^2 \frac{dy dt}{t^{\alpha/\beta}} \\ & \lesssim \int_0^{r^{2\beta}} \left\| (-\Delta)^{1/2} e^{-t(-\Delta)^\beta} \left(\int_0^t (1_{r,x}) v R_1 u dh \right) \right\|^2 \frac{dt}{t^{\alpha/\beta}} \\ & \lesssim r^{2-2\alpha+6\beta-2} \left(\int_0^1 \|M(r^{2\beta} s, \cdot)\|_{L^1(\mathbb{R}^2)} \frac{ds}{s^{\alpha/\beta}} \right) C(\alpha, \beta, f) \\ & \lesssim r^{2-2\alpha+6\beta-2} r^{2-4\beta} r^{2-4\beta} \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta} \\ & \lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}. \end{aligned}$$

Step II: For $j = 1, 2$, R_j are the Riesz transforms $\partial_j (-\Delta)^{-1/2}$. We want to prove:

$$(3.6) \quad r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_j B(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}.$$

Similar to Step I, we can split $B(u, v)$ into $B_i(u, v)$, ($i = 1, 2, 3$). We denote by $A_i, i = 1, 2, 3$

$$(3.7) \quad A_i := r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_j B_i(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}.$$

In order to estimate the term A_1 , we need the following lemma.

Lemma 3.6. For $\beta > 0$, if we denote by K_j^β the kernel of the operator $e^{-t(-\Delta)^\beta} R_j$, we have

$$(1 + |x|)^{n+|\alpha|} \partial^\alpha e^{-t(-\Delta)^\beta} R_j \in L^\infty(\mathbb{R}^2).$$

Proof. By Fourier transform, we have $K_j^\beta = \mathcal{F}^{-1}(\frac{\xi_j}{|\xi|} e^{-|\xi|^{2\beta}})$ where \mathcal{F}^{-1} denotes the inverse Fourier transform. Because

$$\left[\partial^\alpha K_j^\beta(x) \right]^\wedge(\xi) = \frac{\xi_j}{|\xi|} |\xi|^\alpha e^{-|\xi|^{2\beta}} \in L^1(\mathbb{R}^2),$$

we have

$$|\partial^\alpha K_j^\beta(x)| \leq \int_{\mathbb{R}^2} \left| \frac{\xi_j}{|\xi|} |\xi|^\alpha e^{-|\xi|^{2\beta}} \right| d\xi \leq C.$$

Then $\partial^\alpha K_j^\beta(x) \in L^\infty$. If $|x| \leq 1$, we have

$$(1 + |x|)^{n+|\alpha|} |K_j^\beta(x)| \lesssim C_\alpha |K_j^\beta(x)| \lesssim C.$$

If $|x| > 1$, by Littlewood-Paley decomposition and write

$$K_j^\beta(x) = (Id - S_0)K_j^\beta + \sum_{l < 0} \Delta_l K_j^\beta,$$

where $(Id - S_0)K_j^\beta \in \mathcal{S}(\mathbb{R}^2)$ and $\Delta_l K_j^\beta = 2^{2l} \omega_{j,l}(2^l x)$ where $\widehat{\omega_{j,l}}(\xi) = \psi(\xi) \frac{\xi_j}{|\xi|} e^{-|2^l \xi|^{2\beta}} \in L^1(\mathbb{R}^2)$. Then $\omega_{j,l}(x)_{(l < 0)}$ are a bounded set in $\mathcal{S}(\mathbb{R}^2)$. So we have

$$(1 + 2^l |x|)^N 2^{l(2+|\alpha|)} |\partial^\alpha \Delta_l K_j^\beta(x)| \lesssim C_N$$

and

$$\begin{aligned} & |\partial^\alpha S_0 K_j^\beta(x)| \\ & \lesssim C \sum_{2^l |x| \leq 1} 2^{l(2+|\alpha|)} + \sum_{2^l |x| > 1} 2^{l(2+|\alpha|-N)} |x|^{-N} \\ & \lesssim C |x|^{-(2+|\alpha|)}. \end{aligned}$$

Therefore we have completed the proof of Lemma 3.6 \square

Now we complete the proof of Theorem 3.5. In Lemma 3.6, we take $\alpha = 1$ and get

$$\left| \partial_x R_j e^{-t(-\Delta)^\beta}(x, y) \right| \lesssim \frac{1}{(t^{\frac{1}{2\beta}} + |x - y|)^{n+1}}.$$

Similar to the proof in Part I, we can get

$$A_1 := r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y| < r} |R_j B_1(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}.$$

In the proof of Theorem 2.4, we in fact prove the following estimate: for $j = 1, 2$,

$$\begin{aligned} & r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |R_j f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \\ & \lesssim \sup_{r > 0, x_0 \in \mathbb{R}^n} r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0| < r} |f(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}}. \end{aligned}$$

By the above estimate, we have

$$\begin{aligned} A_i &:= r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y| < r} |R_j B_i(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \\ &\lesssim r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y| < r} |B_i(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \end{aligned}$$

where $i = 2, 3$. Following the estimate to $B_i, i = 2, 3$, we can get

$$A_i := r^{2\alpha-2+2\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |R_j B_i(u, v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta}.$$

This completes the proof of Theorem 3.5. \square

Following the method applied in Section 5 of [18], we can easily get the regularity of the solution to the quasi-geostrophic equation (1.3). So we only state the result and leave the proof to the readers. For convenience of the study, we introduce a class of spaces $X_\alpha^{\beta,k}$ as follows.

Definition 3.7. For a nonnegative integer k and $\beta \in (1/2, 1]$, we introduce the space $X_\alpha^{\beta,k}$ which is equipped with the following norm:

$$\|u\|_{X_\alpha^{\beta,k}} = \|u\|_{N_{\alpha,\infty}^{\beta,k}} + \|u\|_{N_{\alpha,C}^{\beta,k}}$$

where

$$\begin{aligned} \|u\|_{N_{\alpha,\infty}^{\beta,k}} &= \sup_{\alpha_1+\dots+\alpha_n=k} \sup_t t^{\frac{2\beta-1+k}{2\beta}} \|\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(\cdot, t)\|_{\dot{B}_{\infty}^{0,1}(\mathbb{R}^n)}, \\ \|u\|_{N_{\alpha,C}^{\beta,k}} &= \sup_{\alpha_1+\dots+\alpha_n=k} \sup_{x_0, r} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |t^{\frac{k}{2\beta}} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2} \\ &\quad + \sum_{j=1}^2 \sup_{\alpha_1+\dots+\alpha_n=k} \sup_{x_0, r} \left(r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j t^{\frac{k}{2\beta}} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2}. \end{aligned}$$

Now we state the regularity result.

Theorem 3.8. Let $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. There exists an $\varepsilon = \varepsilon(2)$ such that if $\|u_0\|_{Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^2)} < \varepsilon$, the solution u to equations (1.3) verifies:

$$t^{\frac{k}{2\beta}} \nabla^k u \in X_\alpha^{\beta,0} \text{ for any } k \geq 0.$$

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